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Two problems of foil consumption optimization for cylindrical bales wrapping

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Abstract. The paper describes two optimization problems for minimization of the consumption of the stretch foil (film) used for wrapping bales to impart bale stability. The first problem consists in the optimal design of foil width. The foil consumption per unit of the bale volume index is used as a measure of foil expenditure. In the second problem a fixed volume of the bale silage must be optimally wrapped by stretch foil and the optimal bale dimensions (diameter, height) are sought out. Mechanical properties of the sealing foil characterized by its Poisson ratio are taken into account. The paper presents optimal and suboptimal solutions to both the problems. The formulas for computing optimal and near-optimal foil width and bale size dimensions are given and estimations of the solution errors are discussed. Simulation results are presented and analyzed for exemplary bale silage.
Key words: baled silage, cylindrical bale, mathematical model, stretch foil consumption, optimization.

INTRODUCTION

One approach to storing agricultural materials such as hay, silage, forage crops, is to package the material in large cylindrical bales. Since its origin in the 1950s the subject of baled silage technique has grown into an area with applications in several branches of agriculture, the number of academics working and patents in the area has increased over the years. A comprehensive review of the studies of bale silage conservation of agricultural materials technique can be found in [1,4,14], and for further research see [5,10,18]. For highlights of progress in silage conservation and future perspectives see, e.g., [21].

The quality of silage in the form of cylindrical and prismatic bales depends, among others, on the efficiency of its protection against the penetration of air and impact of other external factors [18]. Studies concerning the usage of plastic foil to bale wrapping, especially the seal integrity and storage quality depending on different features have been carried out since 1990s, e.g., [1,2,4,6,8,12,20]. Financial expenditures on the purchase of stretch foil constitute a high percentage in the total costs of this technology of silage production [2,15,18]. Although the study of foil usage has been extensive, with conceptual bases supported by empirical data, there are still only a few papers concerning the mathematical description of the foil wrapping process [19,22] and the foil consumption aspects [6,7,11,13,17,18]. The effect of bale size dimensions and the number of foil layers as well as the value of the overlap of the adjacent strips of the foil on the foil consumption has been taken into account for round and square bales in [7,11]. In our previous papers [17,18] the mathematical model that is aimed at an estimation of the foil consumption is developed. A direct analytical formula to compute the final number of wrappings necessary to guarantee the required number of foil layers under the assumed standard of wrapping as a function of initial width of foil, its Poisson's ratio and unit deformation of the foil, bale diameter and the overlap ratio was given, mathematically supported and analysed in detail [17,18]. In result, the mathematical model for exact estimation of the foil consumption for cylindrical bale silages has been derived [18], which serves as a basis for the optimization.

The aim of this study was to find such foil width and bale dimensions for which the consumption of the foil used for wrapping the bale is minimal. To solve these questions, two optimization problems were stated and solved. The optimal foil and bale dimensions were discussed and the problems of suboptimal bale and foil design were covered. The examples of the optimal and suboptimal choice of foil and the bale size parameters were given. The considerations were confined to the widely practiced method of individual wrapping of separate cylindrical bales [4,14], for illustration see Fig. 1 in [18].

MATHEMATICAL MODEL

Here is the mathematical model derived and described in detail in [17,18], that is useful for optimization of the foil consumption.

From the definition of Poisson's ratio v_f we have the following formula [18, Eq. (1)]:

$$b_{fr} = b_f\left(1 - v_f\varepsilon_{lf}\right), \tag{1}$$

which for a given width of non-stretched foil b_f and unit deformation ε_{lf} allows to compute foil width after stretching b_{fr}. As in the previous papers [17,18] we assume that the geometry of movements (the bale's rotation speed and the baler rotation speed) are taken so that the subsequent strips of foil overlap one another creating the overlap $k_f b_{fr}$, where k_f is dimensionless relative ratio determining the width of the contact between adjacent foil strips. This means that all foil strips equally overlap and are overlapped by successive strips. Symmetry of the bale is assumed, thickness of the foil is ignored here (for typical foil and bale dimensions see e.g., [3,11,13,20]). It is also assumed, that the number n_b of a bale rotations around its axis is selected so as to ensure for the taken overlap factor k_f the assumed principal (i.e. minimal on the whole bale surface) number of foil layers. In the examples four layers are considered [12,20]. We assume that the bale is wrapped correctly, when the last applied strip of foil overlaps the preceding strip with overlap $k_f b_{fr}$ and 'overlaps' the first applied foil strip with the overlap not smaller than $k_f b_{fr}$ [17].

It has been proved [17,18] that if the assumed standard of bale wrapping is achieved, then the final number of entire wrappings i_f is uniquely determined by the formula:

$$i_f = \left\lceil \frac{\pi D_b n_b}{b_{fr}(1-k_f)} \right\rceil = \left\lceil \frac{\pi D_b n_b}{b_f(1-v_f\varepsilon_{lf})(1-k_f)} \right\rceil, \tag{2}$$

where: D_b is the outer bale diameter and $\lceil x \rceil$ is the smallest integer not lower than x (ceiling function [9]). The expression under ceiling function brackets in (2) we denote as:

$$i_o = \frac{\pi D_b n_b}{b_{fr}(1-k_f)}. \tag{3}$$

Obviously, the ratio i_o does not have to be (and usually is not) an integer. It provides the lower estimate of i_f, since $i_f = \lceil i_o \rceil \geq i_o$. For the exemplary bale silage the dependence of the final number of wrappings i_f (2) on the width of foil b_f is illustrated in [17; Fig. 4].

FOIL CONSUMPTION OPTIMIZATION

The length of stretched foil L_{fr} wrapped over the bale is equal to:

$$L_{fr} = 2i_f(D_b + H_b), \tag{4}$$

here H_b is bale height, whereas the length of wrapped foil L_f taken from the roll is given by:

$$L_f = \frac{L_{fr}}{\varepsilon_{lf}+1} = \frac{2i_f(D_b+H_b)}{\varepsilon_{lf}+1}. \tag{5}$$

Thus, the surface area of foil taken from the roll $S_f = L_f b_f$ can be directly expressed as:

$$S_f = \frac{2i_f(D_b+H_b)}{\varepsilon_{lf}+1} b_f, \tag{6}$$

where: i_f is given by the right hand side of (2). The dependence of the surface area S_f on the foil width b_f for fixed bale dimensions was studied in the previous paper, see [18, Fig. 3].

A useful measure of the foil consumption is the surface area to volume of silage ratio S_f/V_b [11,17,18], where V_b is the bale volume, which for a cylindrical bale is given by:

$$FC = \frac{S_f}{V_b} = \frac{4S_f}{\pi D_b^2 H_b}. \tag{7}$$

In view of (6) the FC index depends on the number of foil wrappings i_f and including (2) can be rewritten as:

$$FC = \frac{8(D_b+H_b)b_f}{\pi D_b^2 H_b(\varepsilon_{lf}+1)} \left\lceil \frac{\pi D_b n_b}{b_f(1-v_f\varepsilon_{lf})(1-k_f)} \right\rceil. \tag{8}$$

This formula indicates the dependence of the quality index FC on both the mechanical parameters ε_{lf} and v_f of the foil, overlap ratio k_f, number of bale rotations n_b and bale and foil size dimensions D_b, H_b and b_f.

We assume that the overlap k_f and number n_b of bale rotations around bale's axis are adopted in such a way that the pre-assumed principal number of foil layers is guaranteed and that the parameters ε_{lf}, v_f are given. Thus only the bale dimensions H_b, D_b and the width of the foil b_f are decision variables.

OPTIMAL DESIGN OF FOIL WIDTH

Assume the bale diameter D_b is given. The objective is to choose $b_f > 0$ for which the foil consumption described by FC (8) takes the minimal value. The index FC (8) is piecewise increasing function of b_f in the intervals determined by discontinuity points $b_{f,int}$ such that:

$$\frac{\pi D_b n_b}{b_{f,int}(1-\nu_f \varepsilon_{lf})(1-k_f)} = \left\lceil \frac{\pi D_b n_b}{b_{f,int}(1-\nu_f \varepsilon_{lf})(1-k_f)} \right\rceil, \quad (9)$$

i.e., the expression under ceiling function brackets in (9) is integer. In discontinuity points $b_{f,int}$ the lower semi-continuous function $FC(b_f)$ is right-continuous, see Fig. 1. The notation $FC(b_f)$ indicates the dependence of FC given by (8) on the foil width b_f.

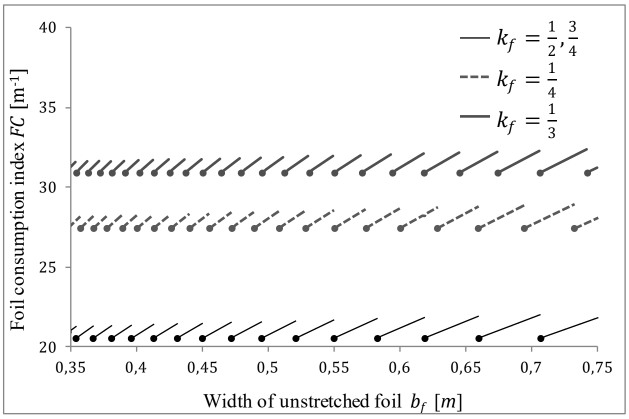

Fig. 1. The foil consumption index $FC(b_f)$ as a function of the width of unstretched foil b_f

Example 1. The following parameters are taken: bale diameter $D_b = 1{,}2\ [m]$ and height $H_b = 1{,}2\ [m]$, Poisson's ratio $\nu_f = 0{,}34\ [-]$ and unit deformation of foil $\varepsilon_{lf} = 0{,}7\ [-]$, which are assumed to be the same for all examples and figures. The overlaps $k_f = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}$ are considered, for which the bale rotation numbers, equal respectively $n_b = 2, 2, 1, \frac{1}{2}$ (see, Table 1) are taken in order to guarantee at least four layers of the foil. The other numerical data are summarized in Table 1 for three widths of non-stretched foil $b_f = 0{,}35; 0{,}5; 0{,}75\ [m]$. The dependence of the function $FC(b_f)$ on the width of foil b_f taken in the interval $0{,}35 \div 0{,}75\ [m]$, cf. [3,11,13] is illustrated in Fig. 1.

In discontinuity points $b_{f,int}$ on the basis of (8) we have:

$$FC(b_{f,int}) = \frac{8 n_b (D_b + H_b)}{D_b H_b (\varepsilon_{lf}+1)(1-\nu_f \varepsilon_{lf})(1-k_f)}, \quad (10)$$

thus the value is, in fact, $b_{f,int}$ independent. Due to the right-continuity of $FC(b_f)$ at discontinuity points, $FC(b_{f,int})$ given by (10) is the minimal value of FC index with respect to foil width b_f, see Fig. 1. Thus, every $b_{f,int}$ defined by equation (9) is global minimum of the function $FC(b_f)$. The following result can be stated.

Corollary 1. Assume the bale diameter D_b is given. Then, the solution of the problem of the foil width design, optimal in the sense of foil consumption index $FC(b_f)$ there exists and is not unique. Every optimal foil width $b_{f,int}$ is uniquely defined by the equation (9). The optimal foil consumption $FC(b_{f,int})$ is given by the right hand side of (10).

The last was, in fact, proved in [17] and signalized in [18], where the results of numerical simulations are presented and discussed. Note, that for any optimal solution $b_{f,int}$ the ratio i_o (3) is integer, a detailed analysis of the wrapping process for integer i_o, which provide some insight into the foil consumption was conducted in [17].

Table 1. Selected numerical data for bale from Example 1

overlap k_f $[-]$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{4}$
bale rotations n_b	2	2	1	$\frac{1}{2}$
number of foil layers	4	4	4	4
width of non-stretched foil b_f $[m]$	0,35	0,35	0,35	0,35
ratio i_o (3)	37,6944	42,4062	28,2708	28,2708
final number of wrappings i_f (2)	38	43	29	29
width of non-stretched foil b_f $[m]$	0,5	0,5	0,5	0,5
ratio i_o (3)	26,38608	29,68434	19,78956	19,78956
final number of wrappings i_f (2)	27	30	20	20
width of non-stretched foil b_f $[m]$	0,75	0,75	0,75	0,75
ratio i_o (3)	17,59072	19,78956	13,19304	13,19304
final number of wrappings i_f (2)	18	20	14	14

SUBOPTIMAL DESIGN OF FOIL WIDTH

The optimal foil consumption is never achieved if (9) is not satisfied. How to choose the practically accessible, i.e. commercially available value of the foil width b_f has been resolved by the next result.

Proposition 1. Assume $\bar{b}_{f,int} > b_{f,int}$ is direct successor of $b_{f,int}$ in the set of foil widths for which the condition (9) is satisfied. Let:

$$b_{f,s} = b_{f,int} + \Delta b_f, \quad (11)$$

be commercially available width of foil such that:

$$b_{f,int} < b_{f,s} < \bar{b}_{f,int}, \quad (12)$$

and assume that $\mathcal{B}_f$ denote the set of all such widths. The foil consumption index $FC(b_f)$ takes the minimal value in the set $\mathcal{B}_f$ if and only if the quotient:

$$\frac{\Delta b_f}{b_{f,int}} = \frac{b_{f,s} - b_{f,int}}{b_{f,int}}, \quad (13)$$

is minimal in the set $\mathcal{B}_f$.

Proof. To prove the above it is enough to see that for $b_{f,s}$ (11) the increase of the foil consumption index:

$$\Delta FC(b_{f,s}) = FC(b_{f,s}) - FC(b_{f,int}), \quad (14)$$

on the basis on (8) and (10) is equal to:

$$\Delta FC(b_{f,s}) = \frac{8(D_b + H_b)}{\pi D_b^2 H_b(\varepsilon_{lf}+1)} \left[b_{f,s} \left\lceil \frac{\pi D_b n_b}{b_{f,s}(1-\nu_f \varepsilon_{lf})(1-k_f)} \right\rceil - \frac{\pi D_b n_b}{(1-\nu_f \varepsilon_{lf})(1-k_f)} \right],$$

and since in view of (12):

$$\left\lceil \frac{\pi D_b n_b}{b_{f,s}(1-\nu_f \varepsilon_{lf})(1-k_f)} \right\rceil = \left\lceil \frac{\pi D_b n_b}{b_{f,int}(1-\nu_f \varepsilon_{lf})(1-k_f)} \right\rceil,$$

including, successively, (11) and next (9) after simple algebraic manipulations we obtain:

$$\Delta FC(b_{f,s}) = \frac{8(D_b + H_b) n_b}{D_b H_b(\varepsilon_{lf}+1)(1-\nu_f \varepsilon_{lf})(1-k_f)} \cdot \frac{\Delta b_f}{b_{f,int}}, \quad (15)$$

whence the validity of the property follows immediately.

Thus, taking this practically acceptable value of the foil width $b_f \in \mathcal{B}_f$ for which $\Delta b_f / b_{f,int}$ is minimal, suboptimal solution can be achieved. This is illustrated by an example.

Example 2. Let us consider again the bale silage from Example 1. Assume $k_f = \frac{1}{2}$. In the interval from 0,35 to 0,75 $[m]$ there are fifteen optimal foil widths $b_{f,int}$ (c.f., Fig. 1). The optimal foil consumption $FC(b_{f,int}) = 20{,}58566209\ [m^{-1}]$. Some of $b_{f,int}$ are summarized in Table 2 together with the suboptimal foil width $b_{f,s} \in \mathcal{B}_f$ being the least integer multiple of 5 centimeter greater than $b_{f,int}$. The respective values of the quotient $\Delta b_f / b_{f,int}$ (13), the increase of the index $\Delta FC(b_{f,s})$ (14) and relative percentage errors of such approximations:

$$ER = \frac{FC(b_{f,s}) - FC(b_{f,int})}{FC(b_{f,int})} 100\%, \quad (16)$$

are also given in Table 2. From a quick inspection of these data it follows, that in the example the best practically realizable foil width is $b_{f,s} = 0{,}4\ [m]$ – the quotient $\Delta b_f / b_{f,int}$ is minimal.

Table 2. The optimal $b_{f,int}$ and suboptimal $b_{f,s}$ widths of non-stretched foil, the quotient $\Delta b_f / b_{f,int}$ (13), the increase of the foil consumption index $\Delta FC(b_{f,s})$ (14) and the relative errors ER (16); the overlap $k_f = 1/2$

$b_{f,int}$ $[m]$	$b_{f,s}$ $[m]$	$\frac{\Delta b_f}{b_{f,int}}$	$\Delta FC(b_{f,s})$	ER [%]
0,366473334	0,40	0,091484599	0,21890563807	1,0633888
0,395791207	**0,40**	**1,40254E-16**	**3,43379E-07**	**1,726E-14**
0,412282506	0,45	0,091484585	0,010859961	0,0527536
0,43020784	0,45	0,046006042	0,010859311	0,0527518
0,449762733	0,45	0,000527539	0,010859731	0,0527539
0,494739001	0,5	0,010633887	0,218905612	1,0633887
0,520777919	0,55	0,056112365	0,010859006	0,0527503
0,549710001	0,55	0,000527549	0,010859939	0,0527549
0,582045897	0,6	0,030846541	0,634996483	3,0846541
0,618423762	0,65	0,051059225	1,051087973	5,1059225
0,659652009	0,7	0,06116557	1,259133774	6,116557
0,706770001	0,75	0,061165583	1,259134015	6,1165583

Summarizing, in the case when any of the exact solutions $b_{f,int}$ of equation (9) is practically realizable, the foil width should be rounded up to that practically acceptable value for which $\Delta b_f / b_{f,int}$ takes minimal value. Different than $b_{f,int}$ foil width means a larger than the optimal $FC(b_{f,int})$ (10) foil consumption. The detailed analysis of (15) show that the deterioration of foil consumption depends, in particular, on the mechanical properties of the foil (Poisson ratio, unit deformation) as well as on the overlap factor k_f and the bale dimensions. The upper bound of the deterioration of foil consumption index is characterized by the next result.

Proposition 2. Let $b_{f,int} < b_f < \bar{b}_{f,int}$, where $\bar{b}_{f,int}$ is direct successor of $b_{f,int}$ in the set of foil widths for which the condition (9) is satisfied. Then, for the loss of foil consumption index the inequality holds:

$$FC(b_f) - FC(b_{f,int}) < \Delta FC_{upp} = FC(b_{f,int})\frac{1}{i_o - 1}, \quad (17)$$

whence for the relative error the following estimation is valid:

$$\frac{FC(b_f) - FC(b_{f,int})}{FC(b_{f,int})} \le \frac{1}{i_o - 1}. \quad (18)$$

Proof. Let us consider minimum $b_{f,int}$ and its direct successor $\bar{b}_{f,int} > b_{f,int}$ in the set of foil widths optimal in the sense of foil consumption index. Since on the basis of (8) and (9) the left-sided limit at discontinuity point $\bar{b}_{f,int}$ exists and is as follows:

$$lim_{b_f \to \bar{b}_{f,int}^-} FC(b_f) = \frac{8(D_b + H_b)n_b}{D_b H_b(\varepsilon_{lf} + 1)(1 - v_f \varepsilon_{lf})(1 - k_f)} \frac{\bar{b}_{f,int}}{b_{f,int}},$$

in view of (10) we obtain:

$$lim_{b_f \to \bar{b}_{f,int}^-} FC(b_f) = FC(b_{f,int})\frac{\bar{b}_{f,int}}{b_{f,int}}. \quad (19)$$

If for $b_{f,int}$ we have (cf. (9) and (3)):

$$\frac{\pi D_b n_b}{b_{f,int}(1 - v_f \varepsilon_{lf})(1 - k_f)} = i_o,$$

where i_o is integer, then for $\bar{b}_{f,int}$ the next equation is satisfied:

$$\frac{\pi D_b n_b}{\bar{b}_{f,int}(1 - v_f \varepsilon_{lf})(1 - k_f)} = i_o - 1.$$

Whence:

$$\frac{\bar{b}_{f,int}}{b_{f,int}} = \frac{i_o}{i_o - 1},$$

and in view of (19) the upper bound of the increase of FC index with respect to its optimal value (foil consumption deterioration) is given by:

$$\Delta FC_{upp} = \lim_{b_f \to \bar{b}_{f,int}^-} FC(b_f) - FC(b_{f,int}) = = FC(b_{f,int})\frac{1}{i_o - 1}. \quad (20)$$

The respective relative error:

$$\frac{\Delta FC_{upp}}{FC(b_{f,int})} = \frac{1}{i_o - 1} \quad (21)$$

is uniquely determined by the ratio i_o, which in the case considered is equal to the number of entire foil wrappings. Now, from (20) and (21) the estimations (17) and (18) immediately follow, and the proof is completed.

Since, the ratio i_o (3) decreases with increasing foil width b_f in view of (18) the increase in foil consumption may accompany the foil width increase (c.f., Fig. 1). The estimation (18) means that, the bigger is the number of foil wrappings, the lower is the maximum growth of the foil consumption. This confirms the results of rough foil consumption analysis from [11] and [18]. In the exemplary bale silage considered here the maximum value of the relative error defined by the left hand side of (18) changes from 3,57% to 7,14% for $k_f = \frac{1}{2}, \frac{3}{4}$ and from 2,38% to 4,55% for $k_f = \frac{1}{3}$. Also Fig. 1 shows the described relations.

DESIGN OF BALE DIMENSIONS

The quality index FC (8) is monotonically decreasing function of the bale height H_b. Simultaneously, FC (8) is a lower semi-continuous function of the bale diameter D_b, which is decreasing function of D_b in the intervals determined by discontinuity points $D_{b,int}$ such that the argument of ceiling function in (8) is integer, see [17; Fig. 6]. Thus FC index cannot be directly applied to foil consumption optimization with respect to D_b and H_b. Additional constraints must be added. Let us consider the following optimization task.

The bale volume $V_b = V_{b0}$ is given. Find geometrical parameters D_b and H_b guarantying the volume V_{b0}, such that index FC takes minimal value. Thus, D_b and H_b are such that:

$$V_{b0} = \frac{\pi D_b^2 H_b}{4}, \quad (22)$$

where the bale size dimensions D_b and H_b are to be determined. From the above we have:

$$H_b = \frac{4V_{b0}}{\pi D_b^2}, \tag{23}$$

and the index FC for given V_{b0} can be rewritten as a function of only the bale diameter D_b described in view of (8) by the function:

$$FC_V(D_b) = \frac{2(\pi D_b^3 + 4V_{b0})b_f}{\pi D_b^2 V_{b0}(\varepsilon_{lf}+1)}\left\lceil \frac{\pi D_b n_b}{b_f(1-v_f\varepsilon_{lf})(1-k_f)} \right\rceil. \tag{24}$$

The exemplary course of $FC_V(D_b)$ as a function of D_b for bale silage from Examples 1,2 and for given $V_{b0} = 1\ [m^3]$ is illustrated for four values of the overlap k_f at Fig. 2.

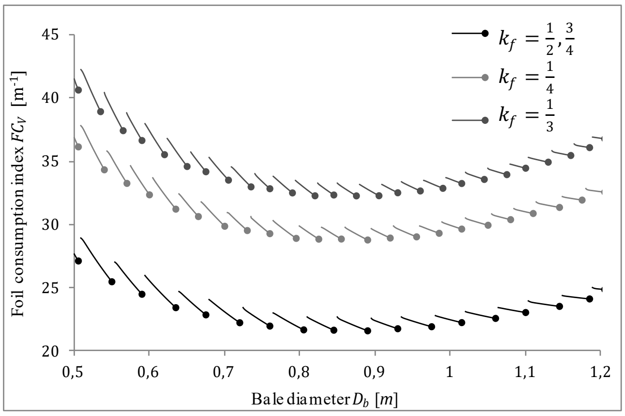

Fig. 2. The foil consumption index $FC_V(D_b)$ for fixed $V_{b0} = 1\ [m^3]$ as a function of the bale diameter D_b for four values of the overlap k_f; $b_f = 0{,}35\ [m]$

Thus the optimization task stated above is equivalent to:

$$FC_V(D_b^*) = min_{D_b>0}\, FC_V(D_b). \tag{25}$$

Similarly to the original index FC (8), the goal function $FC_V(D_b)$ is lower semi-continuous function of the bale diameter D_b, which is decreasing function of D_b in the intervals determined by discontinuity points $D_{b,int}$ such that the argument of ceiling function in (24) is integer, see Fig. 2. To compute the optimal bale dimensions, i.e. to solve the optimization task (25) with non-continuous goal function specific method must be applied. However, first we propose an approach based on the minimization of the lower estimate of foil consumption index.

SUBOPTIMAL DESIGN OF BALE DIMENSIONS

The ceiling function makes the $FC_V(D_b)$ index (24) hard to analyze. One standard method to simplify the analysis is to consider the lower estimate of the rational expression (function) in $\lceil\cdot\rceil$ brackets given by $\lceil x\rceil \geq x$ rather than the exact expression. This approach leads to an approximate value of foil consumption index:

$$FC_{V,l}(D_b) = \frac{\eta}{V_{b0}}\left(D_b^2 + \frac{4V_{b0}}{\pi D_b}\right), \tag{26}$$

which is the lower estimate of $FC_V(D_b)$, where the coefficient:

$$\eta = \frac{2\pi n_b}{(\varepsilon_{lf}+1)(1-v_f\varepsilon_{lf})(1-k_f)}, \tag{27}$$

is introduced for brevity. Note that η is defined only by means of mechanical parameters of the foil and the number of bale rotations, none of them is the subject of the choice in the optimization task. The original function $FC_V(D_b)$ (24) and its lower estimate $FC_{V,l}(D_b)$ (26) are summarized on Fig. 3 for our exemplary bale.

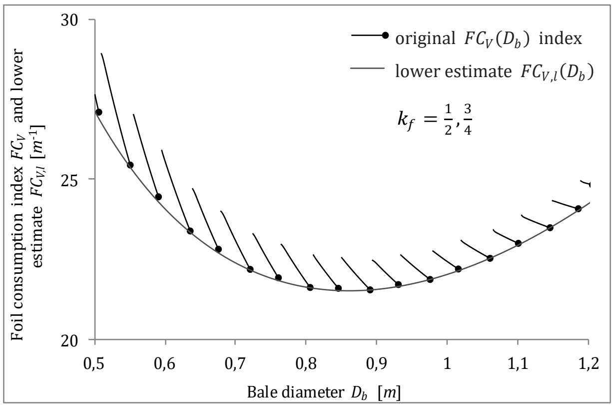

Fig. 3. The foil consumption index $FC_V(D_b)$ and its lower estimate $FC_{V,l}(D_b)$ as a function of the bale diameter D_b; $b_f = 0{,}35\ [m]$, $V_{b0} = 1\ [m^3]$

Note, that $FC_{V,l}(D_b)$ (26) does not depend on the foil width b_f. It is obvious that the two indices $FC_{V,l}(D_b)$ and $FC_V(D_b)$ are equal if and only if the expression under ceiling function brackets in (24) is integer, i.e., whenever the following condition similar to (9) is satisfied:

$$\frac{\pi D_{b,int} n_b}{b_f(1-v_f\varepsilon_{lf})(1-k_f)} = \left\lceil \frac{\pi D_{b,int} n_b}{b_f(1-v_f\varepsilon_{lf})(1-k_f)} \right\rceil. \tag{28}$$

Whence, for any $D_{b,int}$ defined by (28) we have:

$$FC_{V,l}(D_{b,int}) = FC_V(D_{b,int}). \tag{29}$$

Now, the optimization task (25) can be replaced by the problem of finding $\widehat{D}_b$ minimizing $FC_{V,l}(D_b)$ (26) index. Differentiating (26) yields:

$$\frac{dFC_{V,l}(D_b)}{dD_b} = \frac{2\eta}{V_{b0}}\left(D_b - \frac{2V_{b0}}{\pi D_b^2}\right), \quad (30)$$

and next:

$$\frac{d^2FC_{V,l}(D_b)}{dD_b^2} = \frac{4\eta}{V_{b0}}\left(1 + \frac{4V_{b0}}{\pi D_b^3}\right).$$

Since the second derivative is positive for any $D_b > 0$, $FC_{V,l}(D_b)$ is convex function, and from the stationary point condition for unique $\widehat{D}_b$ minimizing $FC_{V,l}(D_b)$ from (30) we obtain:

$$\widehat{D}_b = \sqrt[3]{\frac{2V_{b0}}{\pi}}, \quad (31)$$

whence, in view of (23) and (31), the respective bale height is given by:

$$\widehat{H}_b = 2\sqrt[3]{\frac{2V_{b0}}{\pi}}. \quad (32)$$

This bale dimensions do not depend on the foil width b_f or on the physical properties of the foil or on the overlap ratio k_f, whereas the minimum of lower estimate of optimal foil consumption index:

$$FC_{V,l}(\widehat{D}_b) = \frac{6\eta}{\pi\sqrt[3]{\frac{2V_{b0}}{\pi}}} = \frac{6\eta}{\pi\widehat{D}_b}, \quad (33)$$

depends on the mechanical parameters of foil and overlap ratio k_f, but is b_f independant.

OPTIMAL DESIGN OF BALE DIMENSIONS

Now, how to find the optimal solution of the original optimization task (25) based on $\widehat{D}_b$ is the basic concept. If the factor:

$$\hat{\imath}_o = \frac{\pi\widehat{D}_b n_b}{b_{fr}(1-k_f)} = \frac{\pi\widehat{D}_b n_b}{b_f(1-v_f\varepsilon_{lf})(1-k_f)}, \quad (34)$$

is integer, then $\widehat{D}_b = D_b^*$ and the problem (25) of optimal bale design is solved. Let:

$$\hat{\imath}_f = \lceil\hat{\imath}_o\rceil = \left\lceil\frac{\pi\widehat{D}_b n_b}{b_{fr}(1-k_f)}\right\rceil, \quad (35)$$

be the corresponding final number of entire foil wrappings. If $\hat{\imath}_o$ (34) is non-integer, let us define $\widehat{D}_{b,int}$ such that:

$$\hat{\imath}_f = \frac{\pi\widehat{D}_{b,int}n_b}{b_{fr}(1-k_f)}, \quad (36)$$

whence the following explicit form results:

$$\widehat{D}_{b,int} = \frac{\hat{\imath}_f b_{fr}(1-k_f)}{\pi n_b}. \quad (37)$$

In the considered case of non-integer $\hat{\imath}_o$ (34), the equalities (35) and (36) yields $\widehat{D}_{b,int} > \widehat{D}_b$. Let as also define $\breve{D}_{b,int}$ such that:

$$\breve{D}_{b,int} = \frac{(\hat{\imath}_f - 1)b_{fr}(1-k_f)}{\pi n_b}, \quad (38)$$

for which the next estimation $\breve{D}_{b,int} < \widehat{D}_b$ holds. Thus, in the case of non-integer $\hat{\imath}_o$, diameters $\breve{D}_{b,int}$ and $\widehat{D}_{b,int}$ are lower and upper bounds of $\widehat{D}_b$ according to: $\breve{D}_{b,int} < \widehat{D}_b < \widehat{D}_{b,int}$, and $\widehat{D}_{b,int}$ is, of course, a direct successor of $\breve{D}_{b,int}$ in the set of bale diameters for which the condition (28) is satisfied.

In view of the lower-continuity of the original index $FC_V(D_b)$ (24) and the convexity of the lower bound index $FC_{V,l}(D_b)$, taking into account the equality (29), we conclude that one of the diameters $\breve{D}_{b,int}$ or $\widehat{D}_{b,int}$ or two, simultaneously, is the optimal solution to (25).

Based on (29) we have $FC_V(\widehat{D}_{b,int}) = FC_{V,l}(\widehat{D}_{b,int})$, whence including (26) and (37) after simple algebraic manipulations we obtain:

$$FC_V(\widehat{D}_{b,int}) = \frac{\eta}{V_{b0}}\left[\frac{(\hat{\imath}_f b_{fr})^2(1-k_f)^2}{(\pi n_b)^2} + \frac{4n_b V_{b0}}{\hat{\imath}_f b_{fr}(1-k_f)}\right]. \quad (39)$$

Similarly, (29) implies $FC_V(\breve{D}_{b,int}) = FC_{V,l}(\breve{D}_{b,int})$, and hence by (26) and (38), we have:

$$FC_V(\breve{D}_{b,int}) = \frac{\eta}{V_{b0}}\left[\frac{(\hat{\imath}_f - 1)^2 b_{fr}^2(1-k_f)^2}{(\pi n_b)^2} + \frac{4n_b V_{b0}}{(\hat{\imath}_f - 1)b_{fr}(1-k_f)}\right]. \quad (40)$$

Therefore, in particular, $FC_V(\widehat{D}_{b,int}) < FC_V(\breve{D}_{b,int})$ if and only if the following inequality holds:

$$\frac{(\hat{\imath}_f b_{fr})^2(1-k_f)^2}{(\pi n_b)^2} + \frac{4n_b V_{b0}}{\hat{\imath}_f b_{fr}(1-k_f)} < \frac{(\hat{\imath}_f - 1)^2 b_{fr}^2(1-k_f)^2}{(\pi n_b)^2} + \frac{4n_b V_{b0}}{(\hat{\imath}_f - 1)b_{fr}(1-k_f)},$$

which, after standard algebraic manipulations, can be rewritten in the equivalent form:

$$\hat{\imath}_f(\hat{\imath}_f-1)(\hat{\imath}_f-\tfrac{1}{2})b_{fr}^3(1-k_f)^3<2\pi^2V_{b0}n_b^3.$$

Let us define:

$$\hat{\Delta}=\hat{\imath}_f(\hat{\imath}_f-1)(\hat{\imath}_f-\tfrac{1}{2})b_{fr}^3(1-k_f)^3-2\pi^2V_{b0}n_b^3.$$

Now, we can state the next result.

Theorem 1. Let V_{b0} be the assumed bale volume. If $\lceil\hat{\imath}_o\rceil=\hat{\imath}_o$, i.e., $\hat{\imath}_o$ is integer, then the solution of foil consumption optimization task (25) is unique and such that $D_b^*=\widehat{D}_b$, the minimal value of foil consumption index $FC_V(D_b^*)=FC_{V,l}(\widehat{D}_b)$ is given by the right-hand side of equation (33) and the optimal bale height $H_b^*=\widehat{H}_b=2D_b^*$. If $\hat{\imath}_o$ is non-integer:

(i) if $\hat{\Delta}<0$, then the solution is unique and such that $D_b^*=\widehat{D}_{b,int}$, where $FC_V(D_b^*)=FC_{V,l}(\widehat{D}_{b,int})$ can be computed using formula (26),

(ii) if $\hat{\Delta}>0$, then the solution is unique and such that $D_b^*=\breve{D}_{b,int}$, where $FC_V(D_b^*)=FC_{V,l}(\breve{D}_{b,int})$ can be computed using formula (26),

(iii) If $\hat{\Delta}=0$, then there are two optimal bale diameters $D_b^*=\breve{D}_{b,int}$ and $D_b^*=\widehat{D}_{b,int}$, where the optimal value of foil consumption index is equal to $FC_V(D_b^*)=FC_{V,l}(\breve{D}_{b,int})=FC_{V,l}(\widehat{D}_{b,int})$ and also results from formula (26).

The analysis of case (i) precedes the above result. The formulas for (ii) and (iii) cases can be derived analogue. The optimal in the sense of $FC_V(D_b)$ index bale height can be computed according common to (*i*)–(*iii*) cases general formula, which results directly from (23):

$$H_b^*=\frac{4V_{b0}}{\pi(D_b^*)^2}. \tag{42}$$

ALGORITHM

In view of Theorem 1, the calculation of the values of the bale dimensions minimizing the foil consumption for assumed bale volume V_{b0} and given: foil width b_f, overlap ratio k_f and mechanical parameters of the foil involves the following steps.

1. Determine the suboptimal bale diameter $\widehat{D}_b$ according to the formula (31).
2. Determine the corresponding ratio $\hat{\imath}_o$ (34). If $\hat{\imath}_o$ is integer, then take $D_b^*=\widehat{D}_b$ and compute the minimal value of foil consumption index $FC_V(D_b^*)=FC_{V,l}(\widehat{D}_b)$ by the formula (33). Go to Step 7.
3. Determine the final number of entire foil wrappings $\hat{\imath}_f=\lceil\hat{\imath}_o\rceil$ and, next, determine the index $\hat{\Delta}$ given by (41).
4. If $\hat{\Delta}<0$, then compute $\widehat{D}_{b,int}$ based on (37) and take $D_b^*=\widehat{D}_{b,int}$. Compute minimal value of foil consumption index $FC_V(D_b^*)=FC_V(\widehat{D}_{b,int})$ from (39), or as $FC_V(D_b^*)=FC_{V,l}(\widehat{D}_{b,int})$ using (26). Go to Step 7. (41)
5. If $\hat{\Delta}>0$, then compute $\breve{D}_{b,int}$ based on (38) and take $D_b^*=\breve{D}_{b,int}$. Compute optimal value of foil consumption index $FC_V(D_b^*)=FC_V(\breve{D}_{b,int})$ using (40), or as $FC_V(D_b^*)=FC_{V,l}(\breve{D}_{b,int})$ by applying formula (26). Go to Step 7.
6. If $\hat{\Delta}=0$, then compute $\breve{D}_{b,int}$ based on (38) and, next, $\widehat{D}_{b,int}$ (37) and chose $D_b^*=\widehat{D}_{b,int}$ or $D_b^*=\breve{D}_{b,int}$. Compute optimal value of foil consumption index $FC_V(D_b^*)=FC_V(\widehat{D}_{b,int})=FC_V(\breve{D}_{b,int})$ using (39) or (40). Go to Step 7.
7. Compute the optimal bale height H_b^* according to (42).

The suboptimal $\widehat{D}_b$ (31) not only makes it possible to determine the final number of wrappings $\hat{\imath}_f$ that satisfy the assumed standard of bale wrapping, as characterised by formula (35), but is also significant for the design of algorithm for foil consumption optimization, as it allows to easily determine both the diameters $\widehat{D}_{b,int}$ (37) and $\breve{D}_{b,int}$ (38) as well as the index $\hat{\Delta}$ (41), on the basis of which the optimal solution D_b^* is chosen. However, the determination of $\hat{\Delta}$ (41) enables us to find the optimal bale dimensions without computing both the $\breve{D}_{b,int}$ and $\widehat{D}_{b,int}$ (despite case (iii), only one of them is necessary), since only the final number of entire wrappings $\hat{\imath}_f$ (35) is necessary here. The calculation of $\widehat{D}_b$ is performed prior to its application for estimating $\hat{\imath}_f$, and next $\breve{D}_{b,int}$ or/and $\widehat{D}_{b,int}$ according to the formulas (38) and (37), respectively. The next example illustrates both the optimal and suboptimal approach, where the above algorithm is applied.

Example 3. Let us consider again the bale from previous examples. Assume $k_f=\frac{1}{4},\frac{1}{3},\frac{1}{2}$, $b_f=0{,}35;0{,}5;0{,}75\ [m]$. The bale diameters $\breve{D}_{b,int}$ and $\widehat{D}_{b,int}$, optimal bale dimensions D_b^* and H_b^* and the respective values of the foil consumption index as well as the error:

$$ERR=\frac{FC_V(\widehat{D}_b)-FC_V(D_b^*)}{FC_V(D_b^*)}100\%, \tag{43}$$

are summarized in Table 3 for bale volume $V_{b0}=1\ [m^3]$. The suboptimal bale dimensions computed according to (31) and (32) are b_f and k_f independent, so they are the same for all this data: $\widehat{D}_b=0{,}860254\ [m]$ and $\widehat{H}_b=1{,}720508\ [m]$.

Table 3. Diameters $\breve{D}_{b,int}$ and $\widehat{D}_{b,int}$, optimal D_b^*, H_b^* bale parameters and the values of foil consumption index: suboptimal $FC_V(\widehat{D}_b)$ and optimal $FC_V(D_b^*)$, lower estimate $FC_{V,l}(\widehat{D}_b)$ and errors ERR (43), $V_{b0} = 1\ [m^3]$

$FC_V(\widehat{D}_b)\ [m^{-1}]$	$FC_{V,l}(\widehat{D}_b)[m^{-1}]$	$\breve{D}_{b,int}\ [m]$	$\widehat{D}_{b,int}\ [m]$	$D_b^*\ [m]$	$H_b^*\ [m]$	$FC_V(D_b^*)\ [m^{-1}]$	ERR [%
			$b_f = 0{,}35\ [m],\ k_f = \frac{1}{4}$				
29,75466824	28,715698	0,859544	0,891379	$\breve{D}_{b,int}$	1,723351	28,71572	3,618
			$b_f = 0{,}35\ [m],\ k_f = \frac{1}{3}$				
32,94266841	32,3051603	0,848932	0,8772302	$\breve{D}_{b,int}$	1,7667041	32,31080543	1,956
			$b_f = 0{,}35\ [m],\ k_f = \frac{1}{2}$				
22,31600118	21,53677	0,848932	0,891379	$\breve{D}_{b,int}$	1,7667041	21,540537	3,60
			$b_f = 0{,}5\ [m],\ k_f = \frac{1}{4}$				
28,84381105	28,71569805	0,818613	0,864092	$\widehat{D}_{b,int}$	1,7052583	28,71626792	0,444
			$b_f = 0{,}5\ [m],\ k_f = \frac{1}{3}$				
33,39809701	32,3051603	0,84893246	0,8893578	$\breve{D}_{b,int}$	1,766704164	32,31080543	3,365
			$b_f = 0{,}5\ [m],\ k_f = \frac{1}{2}$				
22,7714298	21,5367735	0,8489324	0,9095705	$\breve{D}_{b,int}$	1,766704164	21,54053696	5,714
			$b_f = 0{,}75\ [m],\ k_f = \frac{1}{4}$				
29,6028587	28,715698	0,8186134	0,886831	$\widehat{D}_{b,int}$	1,61893	28,7425589	2,993
			$b_f = 0{,}75\ [m],\ k_f = \frac{1}{3}$				
34,1571447	32,30516	0,8489325	0,90957	$\breve{D}_{b,int}$	1,766704164	32,31080543	5,714
			$b_f = 0{,}75\ [m],\ k_f = \frac{1}{2}$				
22,7714298	21,5367735	0,818613	0,90957	$\breve{D}_{b,int}$	1,89999460	21,5889463	5,477

Notice, that there is little difference between the optimal D_b^*, suboptimal $\widehat{D}_b$ and $\breve{D}_{b,int}$, $\widehat{D}_{b,int}$ dimensions. The numerical studies have been conducted for bale volume from 1 to 10 $[m^3]$ – the relative error ERR (43) does not exceed 6%.

FINAL REMARKS

Starting with the mathematical model which describes the number of foil wrappings and foil consumption, two problems of the optimal foil as well as bale size dimensions design have been stated, solved and discussed. The mathematical formulas for computing the optimal and near-optimal foil width and bale size dimensions have been derived and estimations of the solution errors given. Computational results have been presented and analyzed for exemplary bale silage. However, such a choice of the foil and bale dimensions that the minimal foil consumption is achieved for square bales is still the open direction of research.

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