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Discrete trigonometric transform and its usage in digital image processing

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Abstract. The achieved discrete trigonometric transform allows building of adaptive filters to suit their area of application. By varying the phase shift can be achieved improved quality of source data recovery in case the frequency coefficients undergo distortion. This transformation is well manifested itself in the problem of image compression, showing better results than the discrete cosine transform.

Key words: discrete cosine transform, discrete Fourier transform, discrete trigonometric transform, signal, filter.

INTRODUCTION

Informatization of the society has led to significant changes in life style [1, 2]. Internet malfunctions can cause a massive panic. At this point (based on 2015), Cisco estimates the total traffic flow of 72,426 exabytes per month [3] and at the same time estimates the growth of Internet traffic by 23% annually. Experts predict that the demand from users in the near future will exceed supply capacity. This is due to the large volumes of traffic generated by growth in popularity services such as YouTube and BBC iPlayer, as well as the fact that more and more people working online. All sorts of experts predict that users will increasingly have to deal with failures in the network, because the free capacity of cyberspace comes to an end. How these apocalyptic predictions come true, it remains to be seen, but it is obvious fact that more urgent task is to filter [4] and compress these demanding data such as images and video. The amount of image information is so large that people fights literally for every byte when it is being compressed, as its storage and transmission costs money. By some estimates additional compression at least 5%, gives a gain in the millions of dollars [5]. Approximately the same situation persists in the transmission of images via the communication channels. The existing capacity is not enough to fully satisfy the needs of users. All of the above determines the relevance of image compression tasks.

THE ANALYSIS OF RECENT RESEARCHES AND PUBLICATIONS

Among the methods of image compression technology most used from JPEG family are JPEG2000 (see, e.g. [6, 7]). At the same time, despite the great efforts aimed at replacing JPEG to JPEG2000, good old JPEG de facto standard for the presentation of images on the Web. This work is devoted JPEG modification in order to increase the compression of images without compromising image quality.

The basis of JPEG is a two-dimensional discrete cosine transform (DCT). Forward transform is given by:

$$D_{\nu,\mu} = \frac{1}{4} C_\nu C_\mu \sum_{j=0}^{N-1}\sum_{j=0}^{N-1} d_{i,j} \cos\left(\frac{(2i+1)\pi\nu}{2N}\right)\cos\left(\frac{(2j+1)\pi\mu}{2N}\right),$$
$$\nu,\mu = 0,1,\ldots,N-1,$$

and inverse transform is given by:

$$d_{i,j} = \frac{1}{4}\sum_{\nu=0}^{N-1}\sum_{\mu=0}^{N-1} C_\nu C_\mu D_{\nu,\mu} \cos\left(\frac{(2i+1)\pi\nu}{2N}\right)\cos\left(\frac{(2j+1)\pi\mu}{2N}\right),$$
$$i,j = 0,1,\ldots,N-1,$$

where:

$$C_i = \begin{cases} 1/\sqrt{N}, i = 0, \\ \sqrt{2/N}, i > 0. \end{cases}$$

Along with the Fourier transform (including cosine transform), a different kind of signal processing using Hartley transform. This conversion is named for R. Hartley, who introduced the integral transform (forward and inverse) in 1942 [8], on the basis of the function;

$$cas\,\theta = \sin\theta + \cos\theta.$$

Interest in the Hartley transform caused by monograph R. Bracewell [9], which have been developed as the foundations of the theory of continuous and discrete Hartley transform.

Interest in Hartley transformation is caused primarily by the fact that in many cases it is advisable to carry out data processing in real numbers, in contrast to the Fourier transform, using a complex exponential.

Forward Hartley transform is implemented as follows:

$$H_i = \sum_{j=0}^{N-1} f_j\, cas\left(\frac{2\pi i}{N} j\right), i = 0,1,\ldots,N-1,$$

and inverse transform is given by:

$$f_i = \sum_{j=0}^{N-1} H_j\, cas\left(\frac{2\pi i}{N} j\right), i = 0,1,\ldots,N-1.$$

It is easy to see that the Hartley transform can be written as:

$$H_i = \sum_{j=0}^{N-1} f_j \cos\left(\frac{2\pi i}{N} j - \varphi\right), i = 0,1,\dots,N-1,$$

where: $\varphi = \dfrac{\pi}{4}$.

Naturally the question arises: whether you can transform for arbitrary φ, and what properties will have this conversion in the case of a positive result. This work is devoted to that issue.

OBJECTIVES

Obtain discrete trigonometric transform that can be used to create adaptive filters for different application areas such as image compressing where that transform produced better results than discrete cosine transform.

THE MAIN RESULTS OF THE RESEARCH

The following assertion holds.

Theorem. Let $\varphi \in \left(0, \dfrac{\pi}{2}\right)$, then for any $\{h_m\}_{m=0}^{N-1}$, where: $-\infty < h_m < \infty, m = 0,1,\dots,N-1$, put

$$H_k = \sum_{m=0}^{N-1} h_m \cos\left(\frac{2\pi mk}{N} - \varphi\right);$$

we have the equality:

$$h_n = \frac{2}{N\sin(2\varphi)} \sum_{k=0}^{N-1} H_k \sin\left(\frac{2\pi nk}{N} + \varphi\right).$$

Proof. We will show that this is indeed the case. Substituting the values of H_k from the first equation into second equation, we obtain:

$$\frac{2}{N\sin(2\varphi)} \sum_{k=0}^{N-1}\left(\sum_{m=0}^{N-1} h_m \cos\left(\frac{2\pi mk}{N} - \varphi\right)\right)\sin\left(\frac{2\pi nk}{N} + \varphi\right) =$$
$$= \frac{2}{N\sin(2\varphi)} \sum_{m=0}^{N-1} h_m \sum_{k=0}^{N-1} \cos\left(\frac{2\pi mk}{N} - \varphi\right)\sin\left(\frac{2\pi nk}{N} + \varphi\right) =$$
$$= \frac{1}{N\sin(2\varphi)} \sum_{m=0}^{N-1} h_m \sum_{k=0}^{N-1}\left(\sin\left(\frac{2\pi k(n+m)}{N}\right) + \sin\left(\frac{2\pi k(n-m)}{N} + 2\varphi\right)\right).$$

Noting that:

$$\sum_{\nu=0}^{N-1} \sin(\alpha + \nu\beta) = \frac{1}{\sin\dfrac{\beta}{2}} \sin\left(\alpha + \frac{N-1}{2}\beta\right)\sin\frac{N\beta}{2}, \quad (1)$$

suppose $n + m \neq N$, then from (1) obtained next equation:

$$\sum_{k=0}^{N-1} \sin\left(\frac{2\pi k(n+m)}{N}\right) =$$
$$\frac{1}{\sin\left(\dfrac{2\pi(n+m)}{2N}\right)} \sin\left(\frac{2\pi(n+m)(N-1)}{2N}\right)\sin\left(\frac{2\pi(n+m)N}{2N}\right).$$

From condition $n+m \neq N$ obtained:

$$\sin\left(\frac{\pi(n+m)}{N}\right) \neq 0,$$

from another side:

$$\sin\left(\frac{2\pi(n+m)N}{2N}\right) = \sin(\pi(n+m)) = 0,$$

and consequently for $n + m \neq N$:

$$\sum_{k=0}^{N-1} \sin\left(\frac{2\pi k(n+m)}{N}\right) = 0 \cdot$$

If $n + m = N$, then:

$$\sum_{k=0}^{N-1} \sin\left(\frac{2\pi k(n+m)}{N}\right) = \sum_{k=0}^{N-1} \sin\left(\frac{2\pi kN}{N}\right) = \sum_{k=0}^{N-1} \sin(2\pi k) = 0.$$

Thus for all $n, m = 0,\dots,N-1$:

$$\sum_{k=0}^{N-1} \sin\left(\frac{2\pi k(n+m)}{N}\right) = 0 \cdot$$

Consider the sum:

$$\sum_{k=0}^{N-1} \sin\left(\frac{2\pi k(n-m)}{N} + 2\varphi\right) \cdot$$

Using (1), we have:

$$\sum_{k=0}^{N-1} \sin\left(\frac{2\pi k(n-m)}{N} + 2\varphi\right) =$$
$$\frac{1}{\sin\left(\dfrac{2\pi(n-m)}{2N}\right)} \sin\left(\frac{2\pi(n-m)(N-1)}{2N} + 2\varphi\right)\sin\left(\frac{2\pi(n-m)N}{2N}\right).$$

Further, we note that since $n, m = 0,\dots,N-1$, then $n - m \neq N$. Now let $n \neq m$, then:

$$\sin\left(\frac{\pi(n-m)}{N}\right) \neq 0,$$

but, at the same time:

$$\sin\left(\frac{2\pi(n-m)N}{2N}\right) = \sin(\pi(n-m)) = 0$$

for all $n, m = 0,\dots,N-1$ and for $n \neq m$.

Finally, let $n = m$, then:

$$\sum_{k=0}^{N-1} \sin\left(\frac{2\pi k(n-m)}{N} + 2\varphi\right) = \sum_{k=0}^{N-1} \sin(0 + 2\varphi) = N\sin(2\varphi) \cdot$$

Thus, we obtain

$$\sum_{k=0}^{N-1}\left(\sin\left(\frac{2\pi k(n+m)}{N}\right) + \sin\left(\frac{2\pi k(n-m)}{N} + 2\varphi\right)\right) =$$

$$\begin{cases} N\sin(2\varphi), & n = m, \\ 0, & n, m = 0,1,\ldots,N-1, n \neq m. \end{cases}$$

Consequently we obtained:

$$\frac{1}{N\sin(2\varphi)} \sum_{m=0}^{N-1} h_m \sum_{k=0}^{N-1}\left(\sin\left(\frac{2\pi k(n+m)}{N}\right) + \sin\left(\frac{2\pi k(n-m)}{N} + 2\varphi\right)\right) =$$

$$\begin{cases} h_n, & m = n, \\ 0, & n, m = 0,1,\ldots,N-1, n \neq m, \end{cases}$$

which completes the proof of the theorem.

Consider some of the properties of the discrete trigonometric transform (DTT). First of all, we note that for $\varphi = \frac{\pi}{4}$ we get a discrete Hartley transform (see e.g. [8])

$$P_i = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} p_k \, cas\left(\frac{2ik\pi}{N}\right),$$

where: $cas\varphi = \cos\varphi + \sin\varphi$.

If

$$F_k = \sum_{m=0}^{N-1} h_m \exp\left(-i\frac{2\pi}{N} mk\right), k = 0,1,\ldots,N-1$$

and

$$h_m = \sum_{k=0}^{N-1} F_k \exp\left(i\frac{2\pi}{N} mk\right), m = 0,1,\ldots,N-1$$

are discrete Fourier transform, then:

$$F_k = F_{k \bmod N} \quad \text{and} \quad F_{N-k} = \overline{F}_k,$$

where: $\overline{z}$ is a complex conjugation of z.

Note that:

$$H_k = \sum_{m=0}^{N-1} h_m \cos\left(\frac{2\pi mk}{N} - \varphi\right) =$$

$$\sum_{m=0}^{N-1} h_m \left(\cos\left(\frac{2\pi mk}{N}\right)\cos(\varphi) + \sin\left(\frac{2\pi mk}{N}\right)\sin(\varphi)\right)$$

Then, we obtained next values:

$$\mathrm{Re}(F_k) = \frac{1}{2\cos\varphi}(H_k + H_{N-k}),$$

$$\mathrm{Im}(F_k) = \frac{1}{2\sin\varphi}(-H_k + H_{N-k}).$$

Note that by varying the phase shift φ we can achieve improvement in the quality of source data recovery in case the frequency coefficients undergo distortion.

Here are a few examples.

Let $N = 4$, then for the original data $\{h_0, h_1, h_2, h_3\}$ we have:

$$H_0 = (h_0 + h_1 + h_2 + h_3)\cos\varphi,$$

$$H_1 = (h_0 - h_2)\cos\varphi + (h_1 - h_3)\sin\varphi,$$

$$H_2 = (h_0 - h_1 + h_2 - h_3)\cos\varphi,$$

$$H_3 = (h_0 - h_2)\cos\varphi - (h_1 - h_3)\sin\varphi.$$

Assume in this case $h_1 \neq h_3$ and $h_0 \neq h_2$ let $H_3 = 0$, then by selecting:

$$\varphi = arctg\left(\frac{h_0 - h_2}{h_1 - h_3}\right), \tag{2}$$

we will obtain a complete data recovery $\{h_0, h_1, h_2, h_3\}$.

Indeed, carrying out the inverse conversion taking into account $H_3 = 0$, we will get:

$$h_0 = \frac{3h_0\cos\varphi + h_1\sin\varphi + h_2\cos\varphi - h_3\sin\varphi}{4\cos\varphi}, \tag{3}$$

$$h_1 = \frac{h_0\cos\varphi + 3h_1\sin\varphi - h_2\cos\varphi + h_3\sin\varphi}{4\sin\varphi}, \tag{4}$$

$$h_2 = \frac{h_0\cos\varphi - h_1\sin\varphi + 3h_2\cos\varphi + h_3\sin\varphi}{4\cos\varphi}, \tag{5}$$

$$h_3 = \frac{-h_0\cos\varphi + h_1\sin\varphi + h_2\cos\varphi + 3h_3\sin\varphi}{4\sin\varphi}. \tag{6}$$

Then, taking into account (2), (3) we will obtain:

$$h_0 = \frac{3h_0\cos\varphi + h_1\sin\varphi + h_2\cos\varphi - h_3\sin\varphi}{4\cos\varphi} =$$

$$= \frac{1}{4}(3h_0 + h_1 tg\varphi + h_2 - h_3 tg\varphi) =$$

$$= \frac{3h_0 + h_2 + (h_1 - h_3)tg\varphi}{4} =$$

$$= \frac{1}{4}\left(3h_0 + h_2 + (h_1 - h_3)\frac{h_0 - h_2}{h_1 - h_3}\right) = h_0.$$

Similarly, by opening of (4) - (5) with (2) we will obtain accurate data reconstruction $\{h_0, h_1, h_2, h_3\}$.

Through the same transform for $N=8$ with $(h_1-h_3+h_5-h_7)(h_0-h_2+h_4-h_6)\neq 0$, let $H_6=0$, then:

$$\varphi = arctg\left(\frac{h_0-h_2+h_4-h_6}{h_1-h_3+h_5-h_7}\right),$$

we will obtain accurate data recovery $\{h_0,h_1,h_2,h_3,h_4,h_5,h_6,h_7\}$.

If $H_7=0$, is true next inequality:

$$\left(h_1+h_2\sqrt{2}+h_3-h_5-h_6\sqrt{2}-h_7\right)\times \left(h_0\sqrt{2}+h_1-h_3-h_4\sqrt{2}-h_5+h_7\right)\neq 0,$$

then:

$$\varphi = arctg\left(\frac{h_0\sqrt{2}+h_1-h_3-h_4\sqrt{2}-h_5+h_7}{h_1+h_2\sqrt{2}+h_3-h_5-h_6\sqrt{2}-h_7}\right).$$

Selecting accurate data recovery $\{h_0,h_1,h_2,h_3,h_4,h_5,h_6,h_7\}$ still will be obtained.

Usage of the possibility of using a phase shift to improve the recovery level allows a construction of adaptive filters based on data. Filter can be for the quantization noise or the nature of the method, which introduces distortion into the signal.

Let's obtain approximate value $\tilde{H}_k\ (k=1,2,\ldots,N-1)$ instead of $H_k\ (k=1,2,\ldots,N-1)$ for selected N.

We will obtain initial data recovery:

$$\tilde{h}_n = \frac{2}{N\sin(2\varphi)}\sum_{k=0}^{N-1}\tilde{H}_k \sin\left(\frac{2\pi nk}{N}+\varphi\right).$$

Error can be calculated by:

$$\varepsilon(\varphi)=\sum_{n=0}^{N-1}\left(h_n-\tilde{h}_n\right)^2 = \sum_{n=0}^{N-1}\left(h_n-\frac{2}{N\sin(2\varphi)}\sum_{k=0}^{N-1}\tilde{H}_k \sin\left(\frac{2\pi nk}{N}+\varphi\right)\right)^2.$$

After that we can find the derivative:

$$\frac{d}{d\varphi}\varepsilon(\varphi)= -\frac{4}{N\sin(2\varphi)}\sum_{n=0}^{N-1}\left(h_n-\frac{2}{N\sin(2\varphi)}\times \sum_{k=0}^{N-1}\tilde{H}_k \sin\left(\frac{2\pi nk}{N}+\varphi\right)\right)\sum_{k=0}^{N-1}\tilde{H}_k \cos\left(\frac{2\pi nk}{N}+\varphi\right).$$

Solving of equation:

$$\sum_{n=0}^{N-1}\left(N\sin(2\varphi)h_n-2\sum_{k=0}^{N-1}\tilde{H}_k \sin\left(\frac{2\pi nk}{N}+\varphi\right)\right)\times \sum_{k=0}^{N-1}\tilde{H}_k \cos\left(\frac{2\pi nk}{N}+\varphi\right)=0,$$

we will find the solution $\varphi_0\in\left(0,\frac{\pi}{2}\right)$, if this is no solution, then the solution lies on the boundary, that is - a discrete cosine transform or the Hartley transform.

Note that processing of a two-dimensional signal, i.e. images are one of the most popular areas of the usage of discrete cosine transform. For example, one of the most popular image compression methods - JPEG based on the use of DCT on squares N×N pixels, where N=8:

$$c_{i,j}^k=\frac{2}{N}C_iC_j\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}p_{n,m}\cos\left(\frac{(2n+1)i\pi}{2N}\right)\cos\left(\frac{(2m+1)j\pi}{2N}\right),$$

where: $C_i=\begin{cases}\frac{1}{\sqrt{2}}, & i=0,\\ 1, & i>0.\end{cases}$

Fairly well-proven two-dimensional discrete Hartley transform:

$$h_{i,j}^k=\frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}p_{n,m}cas\left(\frac{2in\pi}{N}\right)cas\left(\frac{2jm\pi}{N}\right),$$

where: $cas\varphi=\cos\varphi+\sin\varphi$.

The two-dimensional discrete trigonometric transform is given by:

$$h_{i,j}=\frac{2}{N\sin(2\varphi)}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}p_{n,m}\cos\left(\frac{2\pi in}{N}-\varphi\right)\times \cos\left(\frac{2\pi jm}{N}-\psi\right),$$

and inverse transform:

$$p_{n,m}=\frac{2}{N\sin(2\psi)}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}h_{i,j}\sin\left(\frac{2\pi in}{N}+\psi\right)\times \sin\left(\frac{2\pi jm}{N}+\varphi\right).$$

According to the research, numerical experiments were carried out.

As a criterion for evaluating the restoration of the original image $I(i,j)$ by image $K(i,j)$, $(i=0,1,\ldots,m-1,\ j=0,1,\ldots,n-1)$, processed using discrete trigonometric transformation to the quantization of frequency coefficients, used Peak Signal to Noise Ratio:

$$PSNR = 10\log_{10}\left(\frac{MAX_I^2}{MSE}\right) = 20\log_{10}\frac{MAX_I}{\sqrt{MSE}},$$

where:

$$MSE = \frac{1}{nm}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\left|I(i,j) - K(i,j)\right|^2 .$$

For testing were taken images from test base TID2008 by Kodak [10] (see Fig.1).

The graphs (see Fig.2 and Fig.3) show that for all of the typical images the best angles are always lies in the middle of the square:

$$(\varphi,\psi) \in \left(0,\frac{\pi}{4}\right)\times\left(0,\frac{\pi}{4}\right).$$

On the other hand, the Hartley transformation is a special case of the proposed transformation, and DCT is its limiting case.

That is, neither of DCT and DHT are not the best transformation.

Note that using this technology for video compression, we get three free parameters - the angles of phase shifts. Optimizing the angle values for each video scene, we get the opportunity to compress video with higher quality.

Another sphere of application of this transformation is its use for space research, for remote sensing of planets or other space objects.

CONCLUSIONS

On the basis of these results that the use of discrete trigonometric transformation provides a method of image compression JPEG exceeding both the compression ratio and the quality of the reconstructed images.

Thus, the use of the proposed transformation in JPEG instead of DCT allows even for fixed angles from the middle of the square to get a more efficient method of image compression.

If choosing the angles for each particular image, then with the same size of the compressed image, the quality will only increase.

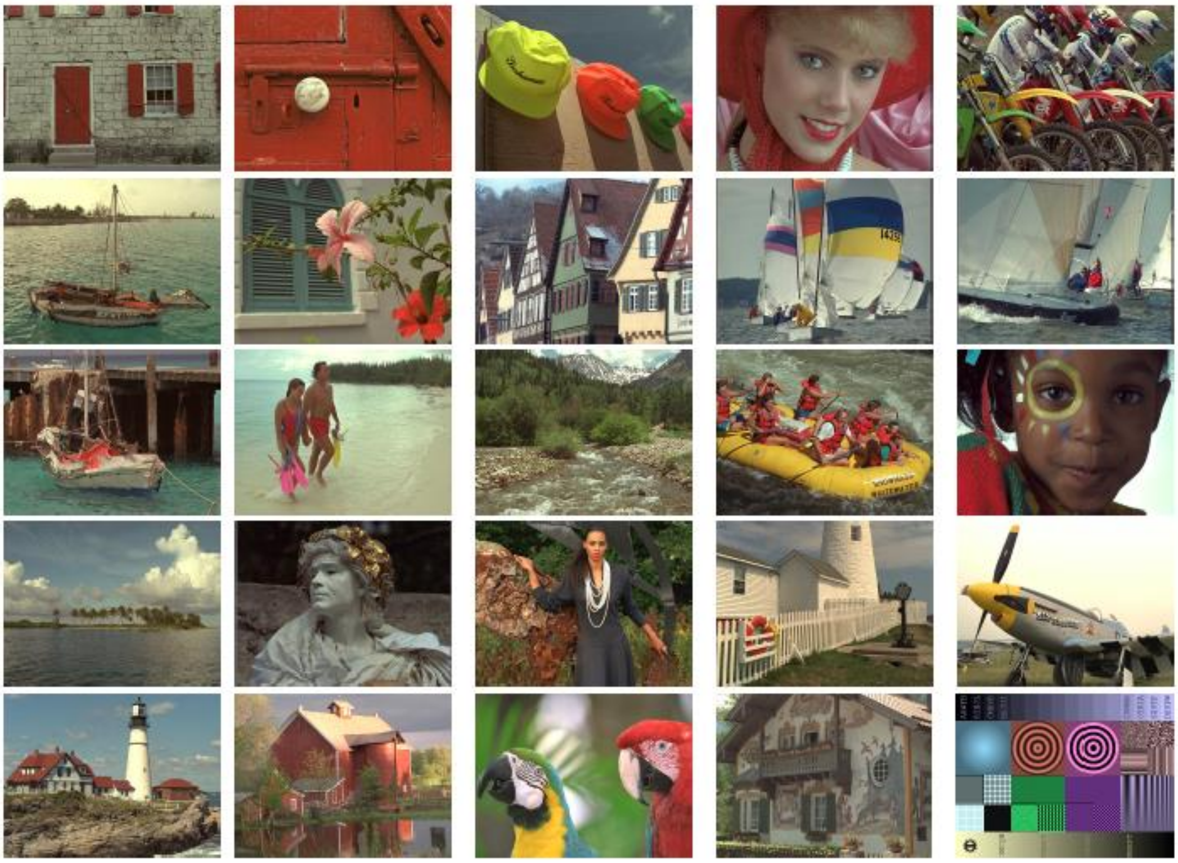

Fig.1. Test images (from test base TID2008)

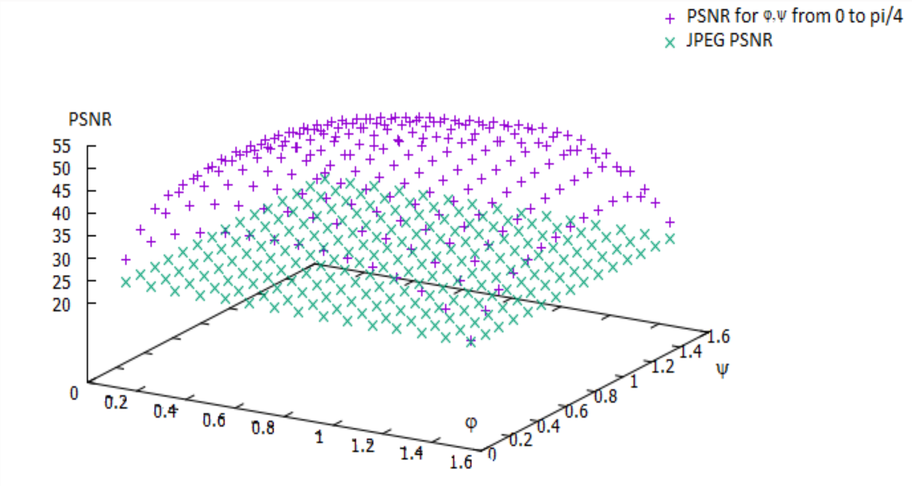

Fig.2. Test results for portrait and landscape images. Quality is higher than JPEG for all values (φ,ψ).

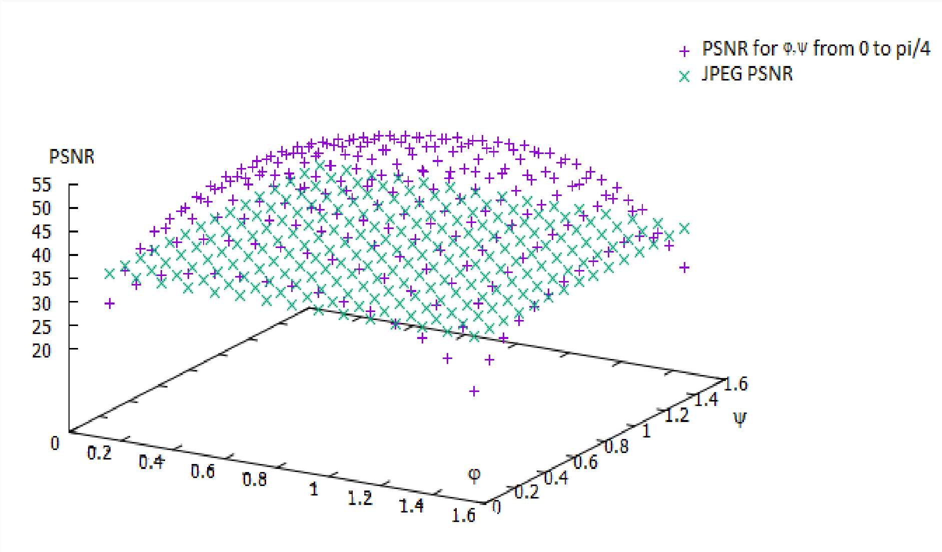

Fig.3. Test results for a synthetic image. Quality is higher than JPEG for (φ,ψ) in the center of the square.

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