ECONTECHMOD. AN INTERNATIONAL QUARTERLY JOURNAL – 2016. Vol. 5. No. 3, 211-216

Optimization of linear functions on a cyclic permutation Based on the random search

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Received July 28.2016: accepted August 30.2016

Abstract. For creating adequate mathematical models of combinatorial problems of constructing optimal cyclic routes, mathematical modeling and solving a number of planning and control tasks solutions of optimization problems on the set of cyclic permutations are required.

Review of the publications on combinatorial optimization demonstrates that the optimization problem on the cyclic permutations have not been studied sufficiently. This paper is devoted to solving optimization problem of a linear function with linear constraints on the set of cyclic permutations. For solving problems of this class using of known methods, taking into account the properties of a combinatorial set of cyclic permutations, is proposed. For this purpose we propose a method based on the ideology of random search. Heuristic method based on the strategy of the branch and bound algorithm is proposed to solve auxiliary optimization problem of a linear function without constraints on the set of cyclic permutations. Since application of the branch and bound algorithm immediately leads to an exponential growth of the complexity with increasing the dimension of the problem a number of modifications are suggested. Modifications allow reducing computational expenses for solving higher dimension problems. The effectiveness of the proposed improvements is demonstrated by computational experiments.

Key words: combinatorial optimization, linear function, cyclic permutations, random search, branch and bound algorithm, parallel computing.

INTRODUCTION

Necessity to solve a wide range of problems in a variety of scientific and applied problems caused the emergence and development of theory and methods for solving problems of linear optimization. Moreover, technology for solving linear programming problems plays a significant role in the creation of algorithms for solving mathematical programming problems of other types, such as combinatorial optimization problems [1, 2].

Mathematical models of many scientific and applied problems of design, planning, placement and control may be adequately represented on the basis of different models of combinatorial optimization [3-6]. In this case variables in these problems are considered as elements of classical combinatorial sets, such as permutations, combinations, arrangements and other [1, 2, 7-9].

Establishing additional constraints on the variables in the combinatorial optimization models leads to appearance of new classes of combinatorial optimization problems. Mathematical models of these new classes of combinatorial optimization problems could be described by subsets of classical combinatorial sets. One of such subsets is set of cyclic permutations [11-13].

There are two known groups of methods for solving combinatorial optimization problems – cutting methods and combinatorial methods [1, 2, 9, 12, 14-17]. One of the most powerful exact methods is combinatorial branch and bounds algorithm [1]. To solve combinatorial optimization problems of large dimension methods based on random search are often used [2, 18].

To improve the efficiency of the known methods of combinatorial optimization the properties of combinatorial sets, describing the admissible area of optimization problems should be used [12, 13, 20].

In this paper, for solving optimization problems on the set of cyclic permutations, an strategy based on the random search, cyclic properties of permutations and analytical solutions of systems of linear inequalities as the constraints on variables are used.

The aim of the paper is elaboration of a strategy for solving optimization problems of linear functions on the set of cyclic permutations with linear constraints.

ANALYSIS OF RECENT RESEARCH AND PUBLICATIONS

Definition 1. A linear ordering of the elements from a certain generating set $\mathrm{A} = \{\mathrm{a}_1, \mathrm{a}_2, ..., \mathrm{a}_n\}$ is called a permutation

$$\pi = \pi(a_1, a_2, ..., a_n) = (\pi(a_1), \pi(a_2), ..., \pi(a_n)) = (a_{i_1}, a_{i_2}, ..., a_{i_n}) = (p_1, p_2, ..., p_n)$$

or, if it is necessary to stress the fact that it contains n elements, n-permutation [15-17].

We denote as P_n the set of permutations generated by the elements $a_1 < a_2 < ... < a_n$.

Consider a certain permutation

$$\pi = (\pi(a_1), \pi(a_2), ..., \pi(a_n)) \in P_n$$

and its element $\pi(a_i) = a_j$, $\forall i, j \in J_n$. Then we can write down:

$$\pi(a_j) = \pi(\pi(a_i)) = \pi^2(a_i).$$

Generally this formula can be represented in the following form:

$$\pi^{k-1}(a_j) = \pi(\pi^{k-1}(a_i)) = \pi^k(a_i),$$
$$\forall i, j \in J_n,\ k \le n.$$

Thus [14], if for some $l \ge 1$ we have $\pi^l(a_i) = a_i$, $i \in J_n$ and all the elements $a_i, \pi(a_i), \pi^2(a_i), ..., \pi^{l-1}(a_i)$ are different, the sequence $(a_i, \pi(a_i), \pi^2(a_i), ..., \pi^{l-1}(a_i))$ is called an l length cycle.

Definition 2. A cyclic permutation is such a permutation π from n elements that contains a single n length cycle, i. e. $\pi^n(a_i) = a_i$, $\forall i \in J_n$ [14]. We denote such permutations as π_C.

Denote P_n^C as set of cyclic permutations without repetitions generated by n elements $a_1 < a_2 < ... < a_n$ [14-17]. Permutation set P_n and set of cyclic permutations P_n^C are Euclidean combinatorial sets, or e-sets [7, 8].

Investigate following combinatorial optimization problem: minimize the linear function with linear constraints on the set of cyclic permutations:

$$L(x) = \sum_{i=1}^{n} \alpha_i p_i \to \min; \qquad (1)$$

$$Cp \le d; \qquad (2)$$

$$p \in P_n^C, \qquad (3)$$

where: $C = [C_{ji}]_{m \times n}$, $d \in R^n$, $\alpha_i \in R$, $p_i \ge 0$;

P_n^C — as set of cyclic permutations without repetitions generated by n real's.

Let us fulfill the enclosure mapping of the permutations set P_n and cyclic permutations P_n^C to the arithmetic Euclidian space R^n. According to [7, 8] the given mapping (which is called immersion) can be represented in the following form:

$$f : P \to R^n,\ \forall p = (p_1, p_2, ..., p_n) \in P,$$
$$x = f(\pi) = (x_1, x_2, ..., x_n) \in E \subset R^n,\ x_i = p_i,$$
$$i \in J_n.$$

As a result of the immersion f we have one-to-one correspondence between each set P_n, P_n^C and

$$E \subset R^n:\ E_n = f(P_n),\ E_n^C = f(P_n^C).$$

Formulate optimization problem equivalent to the problem (1)-(3) using immersion sets P_n, P_n^C into Euclidian space:

$$L(x) = \sum_{i=1}^{n} \alpha_i x_i \to \min; \qquad (4)$$

$$Cx \le d; \qquad (5)$$

$$x \in E_n^C \subset R^n, \qquad (6)$$

where: $C = [C_{ji}]_{m \times n}$, $d \in R^n$, $\alpha_i \in R$, $x_i \ge 0$;

E_n^C – immerse of the cyclic permutations in Euclidean space.

OBJECTIVES

The solution of the problem (4)–(6) is the goal of this work. For reaching this goal the approach based on the random search is used. Properties of permutations immersed in Euclidean space and analytic solution of systems of linear inequalities, describing the task constraints are taken into account. Similar approach earlier was used for optimization of linear function on the set of permutations P_n with linear constraints [12].

Modification of this approach will solve problem (4)–(6) on the set of cyclic permutations. Consider a permutation polyhedron Π_n generated by a set $a_1 < a_2 < ... < a_n$, $vert\ \Pi_n = E_n$ is the set of its vertexes.

Since any cyclic permutation belongs to the set of permutations P_n,

$$\pi_C = (\pi(a_1), \pi(a_2), ..., \pi(a_n)) \in P_n$$

all cyclic permutations are vertexes of the permutation polyhedron Π_n.

Follow the work [12], build n-dimension simplex $T_n \subset R^n$ [18], inclusive polyhedron Π_n. The system of inequalities $C_1 x \le d_1$ defines simplex T_n, where C_1 – matrix of coefficients $(n+1) \times n$, $d_1 \in R^{n+1}$. Since the simplex T_n contains a polyhedron Π_n adding linear inequalities that describe simplex to the constraints (5), all feasible solutions are not changed. System of linear inequalities $W^0 x \le v^0$ combines constraints (5) with inequalities

$$C_1 x \le d_1,$$

where: $W^0 = [w_{ij}^0]$ – $(m+n+1)\times n$-matrix, $v^0 \in R^{m+n+1}$. As a result equivalent optimization problem is constructed:

$$L(x) = \sum_{i=1}^{n} \alpha_i x_i \to \min; \quad (7)$$

$$W^0 x \le v^0; \quad (8)$$

$$x_i \ge 0; \quad (9)$$

$$x \in E_n^C \subset R^n. \quad (10)$$

Consider solution strategy for problem (7)–(10) [12]. In accordance with random search solving process contains M series with m tests in each series.

In each test solution of the system of linear inequalities is founded. According to [20], general formula of non-negative solutions of the system (8)–(9) is:

$$z = \frac{\xi_1 z^1 + \xi_2 z^2 + \ldots + \xi_l z^l}{\xi_1 z_{N+1}^1 + \xi_2 z_{N+1}^2 + \ldots + \xi_l z_{N+1}^l},$$

where: $z^1, z^2, \ldots, z^l$ – fundamental solutions of the following auxiliary system of linear inequalities:

$$\begin{cases} W^0 x - v^0 x_{n+1} \le 0 \\ -x_i \le 0 \end{cases}; \quad (11)$$

where: $z_{n+1}^1, z_{n+1}^2, \ldots, z_{n+1}^l$ – their last coordinates, $\xi_1, \xi_2, \ldots, \xi_l$ – arbitrary real numbers satisfying the condition:

$$\xi_1 z_{n+1}^1 + \xi_2 z_{n+1}^2 + \ldots + \xi_l z_{n+1}^l \ge 0.$$

Next random real numbers $\xi_1, \xi_2, \ldots, \xi_l$, satisfying the condition:

$$\xi_1 z_{n+1}^1 + \xi_2 z_{n+1}^2 + \ldots + \xi_l z_{n+1}^l \ge 0$$

will be generated. Thus will be constructed solution $z(i)$ of the system (11). The nearest to the $z(i)$ point of E_n^C could be found using:

$$x_i = \arg\min_{x \in E_n^C} \|x - z(i)\|^2. \quad (12)$$

If $x_i \in E_n^C$ isn't solution of the system (11), next test begins.

In other way we compare x_i to the previous approximations to problem (7)-(10) solution and if x_i give better value of objective function it will became a new approximation to solution of (7)-(10).

Let $\bar{x} = x_i$. We assume $\bar{x}$ current approximation to solution, and $L(\bar{x}) = \sum_{i=1}^{n} \alpha_i \bar{x}_i$ – upper bound of solution (7)–(10). Add to the system (8) linear inequality:

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n \le \bar{d}, \quad (13)$$

where: $\bar{d} = \sum_{i=1}^{n} \alpha_i \bar{x}_i$.

The result will be $W^1 x \le v^1$ – system of linear inequalities In $W^1 x \le v^1$ there will be inequality with left part, identical to (13): $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n \le \tilde{d}$. Compare $\tilde{d}$ и $\bar{d}$. If $\tilde{d} < \bar{d}$, replace old inequality with the new (13) one in system $W^1 x \le v^1$. It leads to reducing the area restricted with system $W^1 x \le v^1$. Let $\bar{d} = \tilde{d}$. Continue the process.

AUXILIARY PROBLEM

Optimization problem (12) is considered as auxiliary problem.

For the set of permutation solution of this problem is an absolute minimum of linear function [7]:

$$x^* = (x_1^*, x_2^*, \ldots, x_n^*) = \arg\min_{x \in E_n} \sum_{j=1}^{n} c_j x_j, \quad (14)$$

where: $c_j \in R^1$, $\forall j \in J_n$, $x_{m_i}^* = a_i$, $\forall i \in J_n$, and sequence $\{m_1, m_2, \ldots, m_n\}$ such that $c_{m_1} \ge c_{m_2} \ge \ldots \ge c_{m_n}$.

Since set of cyclic permutation is subset of set of permutations for each $x \in E_n^C$ problem (12) can be reduced to problem of linear function optimization.

For combinatorial set of cyclic permutations:

$$\min_{x \in E_n^C} \left(\sum_{j=1}^{n} x_j z(i)_j \right)$$

couldn't be found using simple elements ordering, so we offer to solve this problem using branch and bound algorithm [13].

Evaluation and branching rules are the main components of branch and bound algorithm [1].

In this algorithm branching rule is based on fixing various free generating elements relative the coefficient of the objective function, corresponding to a given level of the tree.

Evaluation of each node of the tree is sum of multiplication of fixed generating elements by the corresponding coefficients of the objective function and multiplication of free generating elements by the coefficients of the objective function according to (14).

The exact solution of the original problem can be obtained using this algorithm. But since solving problem (7)-(10) requires solving auxiliary problem (12) many times for each test in all series the heuristic approach based on branch and bound algorithm was proposed. This approach called up for saving computing resources.

Heuristic solving process consists of two stages. First stage – branch and bound algorithm of finding solution till a user-specified level k. Second stage – final construction of cyclic permutation by random fixing of remaining free generating elements.

Resulting heuristic solution depends on problem dimension n, user-specified level k and on coefficients of the objective function. For example, if user-specified level k almost equal to n, heuristic solution could be equal to exact solution. With decreasing k the probability of coincidence of heuristic and exact solutions decreases too.

PARALLEL COMPUTING

Described strategy for optimization of linear function with linear constraints on the set of cyclic permutations demonstrates good results with relatively small problem dimensions. But increasing n significantly increases time of solving the problem. Features of the strategy give an opportunity for parallel computing.

Solving problem (7)-(10) could be accelerated by parallel solving auxiliary problems for all random generated points in one series. In this case number of tests in one series could be equal to the number of processors.

Thus, strategy of solving problem (7)-(10) using parallel computing consists in such steps:

1) constructing simplex including permutation polyhedron;
2) forming system of constraints;
3) generating number of random points inside solution area;
4) parallel finding nearest vertexes of permutation polyhedron for each random point using branch and bound algorithm or its modification;
5) selection of the best obtained solution;
6) updating the system of constraints.

For evaluating the efficiency of parallel computing Amdahl's and Gustafson's laws are used. They allow to calculate maximum possible acceleration of the parallel computing of the program in comparison with the serial [21].

Let S be acceleration which may be obtained. An estimate for it according to Amdahl's law is:

$$S < \frac{1}{f + \frac{(1-f)}{p}},$$

where: f – the proportion of consecutive operations, p – number of processors.

Also f can be calculated as a percentage of code that couldn't be parallelized. In this case, estimation of S could be less accurate but its calculation is significantly simplified.

Analysis of proposed solutions strategy shows that in our case f is approximately 75%.

The acceleration S:

$$S < \frac{1}{0,75 + \frac{(1-0,75)}{4}} = \frac{1}{0,75 + 0,0625} = 1,23 \cdot$$

MAIN RESULTS OF THE RESEARCH

Described method of optimization of linear functions on the cyclic permutations with linear constraints implemented in software. Coefficients of the objective function and coefficients of constraints were generated randomly. Experiments were carried out in two stages. First stage was dedicated to solving of the problems with dimension of 8 and less variables. These problems were solved with proposed modification of the random search method. The results compared with exact solution obtained by exhaustive search. The coefficients of the objective function generated in the interval [10; 100].

To find the solutions 5 series were used, each series consisted of 10 experimental points. Relative error was calculated for the problems, where approximate solutions was not equal to exact solution. The results are shown in Table 1.

Table 1.

Dimen sion	The number of matching solutions	The number is not matched solutions	Relative error	The average time to solve, s.
3	10	0	0	0,14
4	5	5	0,077	0,269
5	3	7	0,128	0,439
6	4	6	0,072	0,734
7	3	7	0,084	1,527
8	0	10	0,134	3,402

Second stage was dedicated to solving the problems with larger dimensions. For these problems two estimates [13] for the lower estimate of minimum were calculated:

$$E_1 = \left| \frac{Est - Rnd}{Rnd} \right|,$$

$$E_2 = \left| \frac{Est - Rnd}{Est} \right|,$$

where: Est – minimum of objective function on cyclic permutations without linear constraints; Rnd – the solution by random search problem. The results are shown in Table 2.

For the problems with dimension 15 and more auxiliary problem (8) was solved heuristically. For each dimension of the problem tree level k is specified.

To solve the problems, presented in Tables 1 and 2 used 5 series, each consisted of 10 experimental points. It means that for solving each of the problems 50 auxiliary problems were solved.

Table 2.

Dimension	k	Number of tasks	Average estimate E_1	Average estimate E_2	The average time to solve, s.
15	10	10	0,256	0,255	18,19
20	15	10	0,127	0,122	225,9
25	15	3	1,716	0,44	809,17

To reduce the time and technical resources, were conducted experiments with 2 series, each consisted of 15 experimental points. The results are shown in Table 3.

Table 3.

Dimension	k	Number of tasks	Average estimate E_1	Average estimate E_2	The average time to solve, s.
25	15	7	0,1652	0,203	821,14
30	15	3	0,227	0,308	1805,86
35	15	1	0,219	0,2819	11215,516
40	20	1	0,305	0,379	33112,95

Experiments to evaluate efficiency of the parallel computing of problem (4)-(7) were conducted. The real acceleration of the parallel algorithm is the ratio of the execution time of sequential algorithm to the time of parallel algorithm execution: $S = \frac{T_1}{T_p}$, where T_1 – runtime of sequential algorithm, T_p – runtime using p processors.

To evaluate the scalability of the parallel algorithm the concept of the coefficient of efficiency of parallelization is used: $E = \frac{S}{p}$.

Table 4.

Dimension	The time to solve, T_1, s.	The time to solve $T_p = T_4$, s.	$S = \frac{T_1}{T_p}$	$E = \frac{S}{p}$
15	493,7	281,5	1,75	0,4375
20	4782,9	3297,5	1,45	0,3625
25	62559,9	31688,3	1,97	0,4925

Note, that actual acceleration obtained by using parallel computing, more than the theoretical evaluation by Amdahl's Law. This is associated with inexact rough estimate of f.

CONCLUSIONS

This paper suggests a strategy for solving a combinatorial optimization problems with linear objective function and linear constraints on the set of cyclic permutations. This strategy is based on the random search, cyclic properties of permutations and analytical solutions of systems of linear inequalities as the constraints on variables.

The solving the auxiliary problem of combinatorial optimization on the set of cyclic permutations without restriction performed using the branch and bound method. It should be noted that the use of branch and bound algorithm leads to an exponential increase in the complexity of solution with increasing dimension of the problem. Moreover, rule of branching and the rule of estimates calculating significantly depend on the complexity of solving process. The following modifications of the method are proposed to decrease the complexity:

1. heuristic method for solving the auxiliary problem;
2. modification with the parallel computing for the auxiliary problem.

These modifications allow reducing the computational expences for large-scale problems. The effectiveness of the proposed improvements is confirmed by computational experiments.

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